A class of groups for which every action is W*-superrigid

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Abstract

We prove the uniqueness of the group measure space Cartan subalgebra in crossed products $A \rtimes \Gamma$ covering certain cases where Γ is an amalgamated free product over a non-amenable subgroup. In combination with Kida's work we deduce that if $\Sigma < \mathrm{SL}(3,\mathbb{Z})$ denotes the subgroup of matrices g with $g_{31} = g_{32} = 0$, then any free ergodic probability measure preserving action of $\Gamma = \mathrm{SL}(3,\mathbb{Z}) *_\Sigma \mathrm{SL}(3,\mathbb{Z})$ is stably W*-superrigid. In the second part we settle a technical issue about the unitary conjugacy of group measure space Cartan subalgebras.

1 Introduction

This short article is a two-fold complement to [PV09]. The main result of [PV09] provides a class $\mathcal{G}$ of groups Γ such that for every free ergodic probability measure preserving (pmp) action $\Gamma \curvearrowright (X,\mu)$, the II_1 factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ has a unique group measure space Cartan subalgebra up to unitary conjugacy. The class $\mathcal{G}$ contains all non-trivial amalgamated free products $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ such that Γ admits a non-amenable subgroup with the relative property (T) and such that Σ is an amenable group that is sufficiently non normal in Γ. In combination with known orbit equivalence superrigidity theorems, several group actions $\Gamma \curvearrowright (X,\mu)$ are shown in [PV09] to be W*-superrigid: any isomorphism between $\mathrm{L}^\infty(X) \rtimes \Gamma$ and an arbitrary group measure space II_1 factor $\mathrm{L}^\infty(Y) \rtimes \Lambda$, comes from a conjugacy of the actions. For example the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$ is W*-superrigid for many of the groups $\Gamma \in \mathcal{G}$, see [PV09, Theorem 1.3]. Using Kida's [Ki09, Theorem 1.4] and denoting by $\Sigma < \mathrm{SL}(3,\mathbb{Z})$ the subgroup of upper triangular matrices, one deduces W*-superrigidity for every free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ such that all finite index subgroups of Σ still act ergodically, see [PV09, Theorem 6.2].

In the first part of this article we generalize the uniqueness of the group measure space Cartan subalgebra to the case where $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is an amalgamated free product over a possibly non-amenable subgroup Σ, see Theorem 5. We still assume some softness on Σ by imposing the existence of a normal tower $\{e\} = \Sigma_0 \lhd \Sigma_1 \lhd \cdots \lhd \Sigma_{n-1} \lhd \Sigma_n = \Sigma$ such that all quotients Σ_i/Σ_{i-1} have the Haagerup property. We have to strengthen however the rigidity assumption by imposing that Γ admits an infinite subgroup that has property (T). The proof of Theorem 5 is identical to the proof of [PV09, Theorem 5.2], apart from the fact that we need a new transfer of rigidity lemma, see Lemma 4 (cf. [PV09, Lemma 3.1]).

Using Kida's [Ki09, Theorem 9.11] it follows that if $\Sigma < \mathrm{SL}(3,\mathbb{Z})$ denotes the subgroup of matrices g with $g_{31} = g_{32} = 0$, then any free ergodic pmp action of Γ on (X,μ) is stably W*-superrigid, see Theorem 3. Contrary to the case where Σ consists of the upper triangular matrices, no ergodicity assumption has to be made on the action of the finite index subgroups of Σ.

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In the second part of this article we provide a detailed argument for the following principle: if $B \subset A \rtimes \Gamma$ is a Cartan subalgebra in a group measure space II_1 factor and if B embeds into $A \rtimes \Sigma$ for a sufficiently non normal subgroup $\Sigma < \Gamma$, then B and A are unitarily conjugate. So Proposition 8 provides a justification for the end of the proofs of [PV09, Theorems 5.2 and 1.4], which were rather brief compared to the rest of that article. We are very grateful to Steven Deprez who pointed out to us the necessity of adding more details.

2 Preliminaries

Intertwining-by-bimodules

Let (M, τ) be a von Neumann algebra with a faithful normal tracial state τ and let $A, B \subset M$ be (possibly non-unital) von Neumann subalgebras. In [Po03, Section 2] the technique of intertwining-by-bimodules was introduced. It is shown there that the following two conditions are equivalent.

- There exist projections $p \in A$, $q \in B$, a non-zero partial isometry $v \in pMq$ and a normal $*$-homomorphism $\theta : pAp \to qBq$ satisfying $xv = v\theta(x)$ for all $x \in pAp$.
- There is no sequence of unitaries (w_n) in A such that $\|E_B(aw_nb)\|_2 \to 0$ for all $a, b \in M$.

If one, and hence both, of these conditions hold, we write $A \prec_M B$. By [Po01, Theorem A.1], if A and B are Cartan subalgebras of a II_1 factor M, then $A \prec_M B$ if and only if A and B are unitarily conjugate.

Property (T) for von Neumann algebras

Let (P, τ) be a von Neumann algebra with a faithful normal tracial state τ. A normal completely positive map $\varphi : P \to P$ is said to be subunital if $\varphi(1) \leq 1$ and subtracial if $\tau \circ \varphi \leq \tau$. Following [CJ85], we say that P has property (T) if every sequence of normal subunital subtracial completely positive maps $\varphi_n : P \to P$ converging to the identity pointwise in $\| \cdot \|_2$, converges uniformly in $\| \cdot \|_2$ on the unit ball of P.

If Γ is a countable group, then $\mathrm{L}\,\Gamma$ has property (T) if and only if the group Γ has property (T) in the usual sense.

Relative property (H) and property anti-(T)

In [Po01, Section 2] property (H) of a finite von Neumann algebra M relative to a von Neumann subalgebra $P \subset M$ is introduced. We recall from [Po01] the following facts.

- If $\Gamma \curvearrowright P$ is a trace preserving action, then $P \rtimes \Gamma$ has property (H) relative to P if and only if the group Γ has the Haagerup property. Recall that a countable group Γ has the Haagerup property if and only if there exists a sequence of positive definite functions $\varphi_n : \Gamma \to \mathbb{C}$ tending to 1 pointwise and such that for every n the function φ_n belongs to $c_0(\Gamma)$ (see e.g. [CCJJV]).

- If M has property (H) relative to $P \subset M$, there exists a sequence of normal subunital subtracial completely positive P-bimodular maps $\varphi_n : M \to M$ such that $\|\varphi_n(x) - x\|_2 \to 0$ for every $x \in M$ and such that every φ_n satisfies the following relative compactness property: if (w_k) is a sequence of unitaries in M satisfying $\|E_P(aw_kb)\|_2 \to 0$ for all $a, b \in M$, then $\|\varphi_n(w_k)\|_2 \to 0$ when $k \to \infty$. The converse is almost true, but we have no need to go into these technical details.

- If M has property (H) relative to $P \subset M$, then $N \overline{\otimes} M$ has property (H) relative to $N \overline{\otimes} P$ for every finite von Neumann algebra N.

The following lemma is essentially contained in [Po01, Theorem 6.2]. We provide a full proof for the convenience of the reader.

Lemma 1. *Let (M, τ) be a tracial von Neumann algebra and $P_1 \subset P \subset M$ von Neumann subalgebras. Assume that P has property (H) relative to P_1.*

If $M_0 \subset pMp$ is a von Neumann subalgebra such that M_0 has property (T) and $M_0 \prec_M P$, then $M_0 \prec_M P_1$.

Proof. Assume that $M_0 \not\prec_M P_1$. Since $M_0 \prec_M P$, by [Va07, Remark 3.8] we find non-zero projections $p_0 \in M_0$, $q \in P$, a non-zero partial isometry $v \in p_0 M q$ and a normal unital $*$-homomorphism $\theta : p_0 M_0 p_0 \to qPq$ such that $xv = v\theta(x)$ for all $x \in p_0 M_0 p_0$ and such that $\theta(p_0 M_0 p_0) \not\prec_P P_1$.

Since P has property (H) relative to P_1, we can take a sequence $\varphi_n : P \to P$ of normal subunital subtracial completely positive maps such that $\|\varphi_n(x) - x\|_2 \to 0$ for all $x \in P$ and such that every φ_n satisfies the relative compactness property explained above. Since $\theta(p_0 M_0 p_0)$ has property (T), take n such that $\|\varphi_n(w) - w\|_2 \leq \|q\|_2/2$ for all unitaries $w \in \theta(p_0 M_0 p_0)$. Since $\theta(p_0 M_0 p_0) \not\prec_P P_1$, take a sequence of unitaries $w_k \in \theta(p_0 M_0 p_0)$ such that $\|E_{P_1}(aw_kb)\|_2 \to 0$ for all $a, b \in P$. By the relative compactness of φ_n, it follows that $\|\varphi_n(w_k)\|_2 \to 0$ when $k \to \infty$. So, for k large enough, we have $\|\varphi_n(w_k)\|_2 < \|q\|_2/2$. It follows that

$$\|q\|_2 = \|w_k\|_2 \leq \|\varphi_n(w_k)\|_2 + \|\varphi_n(w_k) - w_k\|_2 < \|q\|_2 ,$$

which is absurd. □

We say that a finite von Neumann algebra P is *anti-(T)* if there exist von Neumann subalgebras $\mathbb{C}1 = P_0 \subset P_1 \subset \cdots \subset P_n = P$ such that for all $i = 1, \ldots, n$, the von Neumann algebra P_i has property (H) relative to P_{i-1}. Repeatedly applying Lemma 1, an anti-(T) von Neumann algebra cannot contain a diffuse von Neumann subalgebra with property (T).

We say that a countable group Σ is anti-(T) if there exist subgroups $\{e\} = \Sigma_0 < \Sigma_1 < \cdots < \Sigma_n = \Sigma$ such that for all $i = 1, \ldots, n$, Σ_{i-1} is normal in Σ_i and Σ_i/Σ_{i-1} has the Haagerup property. If Σ is anti-(T), then the group von Neumann algebra $\mathrm{L}\,\Sigma$ is anti-(T) as well. An anti-(T) group cannot contain an infinite subgroup with property (T). Nevertheless, $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ is an anti-(T) group (since $\mathrm{SL}(2, \mathbb{Z})$ has the Haagerup property and $\mathbb{Z}^2$ is amenable) which contains an infinite subgroup with the *relative* property (T), namely $\mathbb{Z}^2$. This explains why our new transfer of rigidity lemma (see Lemma 4) requires property (T) rather than relative property (T).

3 Transfer of rigidity and W*-superrigidity

We say that free ergodic pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\eta)$ are

- W*-equivalent, if $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$,
- orbit equivalent, if the orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\Lambda \curvearrowright Y)$ are isomorphic,
- conjugate, if there exists an isomorphism of probability spaces $\Delta : X \to Y$ and an isomorphism of groups $\delta : \Gamma \to \Lambda$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ almost everywhere.

Following [PV09, Definition 6.1] a free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ is called *W*-superrigid* if the following property holds. If $\Lambda \curvearrowright (Y,\eta)$ is an arbitrary free ergodic pmp action and $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ is a W*-equivalence, then the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate through $\Delta : X \to Y$, $\delta : \Gamma \to \Lambda$ and up to unitary conjugacy π is of the form

$$\pi(au_g) = \Delta_*(a\omega_g)u_{\delta(g)} \quad \text{for all} \quad a \in L^\infty(X), g \in \Gamma,$$

where $(\omega_g) \in Z^1(\Gamma \curvearrowright X)$ is a $\mathbb{T}$-valued 1-cocycle for the action $\Gamma \curvearrowright X$.

Slightly more natural than W*-superrigidity is the notion of *stable W*-superrigidity* where possible finite index issues are correctly taken into account. A stable isomorphism between II_1 factors M and N is an isomorphism between M and an amplification N^t. This leads to the notion of *stable W*-equivalence* between free ergodic pmp actions. Similarly one defines *stable orbit equivalence.* Finally, a *stable conjugacy* between two free ergodic pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\eta)$ is a conjugacy between the actions $\frac{\Gamma_0}{G} \curvearrowright \frac{X_0}{G}$ and $\frac{\Lambda_0}{H} \curvearrowright \frac{Y_0}{H}$ where $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ are induced(4) from $\Gamma_0 \curvearrowright X_0$, $\Lambda_0 \curvearrowright Y_0$ and where $G \lhd \Gamma_0$, $H \lhd \Lambda_0$ are finite normal subgroups.

Definition 2 ([PV09, Definition 6.4])**.** A free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ is said to be *stably W*-superrigid* if the following holds. Whenever π is a stable W*-equivalence between $\Gamma \curvearrowright (X,\mu)$ and an arbitrary free ergodic pmp action $\Lambda \curvearrowright (Y,\eta)$, the actions are stably conjugate and π equals the composition of

- the canonical stable W*-equivalence given by the stable conjugacy,
- the automorphism of $L^\infty(X) \rtimes \Gamma$ given by an element of $Z^1(\Gamma \curvearrowright X)$,
- an inner automorphism of $L^\infty(X) \rtimes \Gamma$.

Let $\Gamma \curvearrowright (X,\mu)$ be stably W*-superrigid. If moreover Γ has no finite normal subgroups and if finite index subgroups of Γ still act ergodically on (X,μ), then $\Gamma \curvearrowright (X,\mu)$ is W*-superrigid in the sense explained above.

The following is the main result in this section.

Theorem 3. *Denote by $\Sigma < SL(3,\mathbb{Z})$ the subgroup of matrices g such that $g_{31} = g_{32} = 0$. Put $\Gamma = SL(3,\mathbb{Z}) *_\Sigma SL(3,\mathbb{Z})$. Every free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ is stably W*-superrigid. In particular all a-periodic(5) free ergodic pmp actions of Γ are W*-superrigid.*

(4) A free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ is said to be induced from $\Gamma_0 \curvearrowright X_0$ if $\Gamma_0 < \Gamma$ is a finite index subgroup and $X_0 \subset X$ is a non-negligible Γ_0-invariant subset such that $\mu(X_0 \cap g \cdot X_0) = 0$ for all $g \in \Gamma - \Gamma_0$.

(5) A free ergodic pmp action is called a-periodic if it is not induced from a finite index subgroup, i.e. if finite index subgroups still act ergodically.

More generally, if $k \geq 1$ and $n_1, \dots, n_k \in \{1,2\}$, the same conclusion holds for the group $\Gamma = \mathrm{PSL}(n,\mathbb{Z}) *_\Sigma \mathrm{PSL}(n,\mathbb{Z})$ where $n = 2 + n_1 + \cdots + n_k$ and Σ is the image in $\mathrm{PSL}(n,\mathbb{Z})$ of

$$\mathrm{SL}(n,\mathbb{Z}) \cap \begin{pmatrix} \mathrm{GL}(2,\mathbb{Z}) & * & \cdots & * \\ 0 & \mathrm{GL}(n_1,\mathbb{Z}) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_k,\mathbb{Z}) \end{pmatrix}.$$

Proof. The theorem is a direct consequence of Kida's [Ki09, Theorem 9.11] and the uniqueness of group measure space Cartan theorem 5 below. Let

$$P = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \text{ and } G = \mathrm{SL}(n,\mathbb{Z}) \cap \begin{pmatrix} \mathrm{GL}(2,\mathbb{Z}) & 0 & \cdots & 0 \\ 0 & \mathrm{GL}(n_1,\mathbb{Z}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_k,\mathbb{Z}) \end{pmatrix},$$

and denote by $\overline{P}$ (resp. $\overline{G}$) the image of P (resp. G) in $\mathrm{PSL}(n,\mathbb{Z})$. We have that $\overline{P}$ is amenable and normal in Σ and $\overline{G} \cong \Sigma/\overline{P}$ has the Haagerup property. This shows that Σ is anti-(T). Therefore, if $\Gamma \curvearrowright (X,\mu)$ is an arbitrary free ergodic pmp action, Theorem 5 says that every stable W*-equivalence between $\Gamma \curvearrowright (X,\mu)$ and an arbitrary $\Lambda \curvearrowright (Y,\eta)$ comes from a stable orbit equivalence of the actions. Kida showed in [Ki09, Theorem 9.11] that Γ is coupling rigid with respect to the abstract commensurator[(6)] $\mathrm{Comm}(\Gamma)$. Since Γ is icc and $\mathrm{Comm}(\Gamma)$ is countable, this precisely means that every stable orbit equivalence comes from a stable conjugacy, cf. [Ki09, Proposition 3.11]. ☐

The W*-superrigidity in Theorem 3 arises as the combination of Kida's OE superrigidity and the following uniqueness result for group measure space Cartan subalgebras. We first need a new transfer of rigidity lemma (cf. [PV09, Lemma 3.1]).

Lemma 4. *Let M be a II_1 factor and $\varphi_n : M \to M$ a sequence of normal subunital subtracial completely positive maps. Assume that $P, M_0 \subset M$ are von Neumann subalgebras such that P is anti-(T) and such that M_0 is diffuse and has property (T).*

Let $p \in M$ be a projection and $pMp = Q \rtimes \Lambda$ any crossed product decomposition with Q being anti-(T). Denote by $(v_s)_{s\in\Lambda}$ the corresponding canonical unitaries.

For every $\varepsilon > 0$, there exists n and a sequence $(s_k)_{k\in\mathbb{N}}$ in Λ such that

1. *$\|\varphi_n(v_{s_k}) - v_{s_k}\|_2 \leq \varepsilon$ for all $k \in \mathbb{N}$,*
2. *for all $a, b \in M$ we have $\|E_P(a v_{s_k} b)\|_2 \to 0$ when $k \to \infty$.*

Proof. Since M is a II_1 factor and M_0 is diffuse, we may assume that $p \in M_0$. Write $N = Q \rtimes \Lambda$ and denote by $\Delta : N \to N \overline{\otimes} N$ the normal $*$-homomorphism given by $\Delta(a v_s) = a v_s \otimes v_s$ for all $a \in Q$ and $s \in \Lambda$. Normalize the trace τ on M such that $\tau(p) = 1$.

(6) Given a group Γ the abstract commensurator $\mathrm{Comm}(\Gamma)$ is defined as the group of all isomorphisms $\delta : \Gamma_1 \to \Gamma_2$ between finite index subgroups $\Gamma_1, \Gamma_2 < \Gamma$, identifying two such isomorphisms when they coincide on a finite index subgroup. Inner conjugacy provides a homomorphism from Γ to $\mathrm{Comm}(\Gamma)$, which is injective if and only if Γ is icc.

Since Q is anti-(T), Lemma 1 implies that $pM_0p \not\prec_N Q$. Let (w_n) be a sequence of unitaries in pM_0p such that $\|E_Q(aw_nb)\|_2 \to 0$ for all $a, b \in N$. It follows that $\Delta(w_n)$ is a sequence of unitaries in $N \overline{\otimes} N$ satisfying

$$\|(\mathrm{id} \otimes \tau)(a\Delta(w_n)b)\|_2 \to 0 \quad \text{for all} \quad a, b \in N \overline{\otimes} N \; .$$

Indeed, it suffices to check the convergence for $a = 1 \otimes cv_s$ and $b = 1 \otimes dv_t$, where $c, d \in Q$ and $s, t \in \Lambda$. We have $\|(\mathrm{id} \otimes \tau)((1 \otimes cv_s)\Delta(w_n)(1 \otimes dv_t))\|_2 = |\tau(\sigma_{t^{-1}}(d)c)| \|E_Q(w_n v_{ts})\|_2 \to 0$. So, $\Delta(pM_0p) \not\prec_{N \overline{\otimes} N} N \otimes 1$. Since P is anti-(T), Lemma 1 implies that $\Delta(pM_0p) \not\prec_{N \overline{\otimes} M} N \overline{\otimes} P$.

Choose $\varepsilon > 0$. Put $\varepsilon_1 = \varepsilon^2/4$. Since pM_0p has property (T), take n such that

$$1 - \mathrm{Re}(\tau \otimes \tau)(\Delta(w)^* \, (\mathrm{id} \otimes \varphi_n)\Delta(w)) \leq \varepsilon_1$$

for all $w \in \mathcal{U}(pM_0p)$. Define

$$\mathcal{V} := \{ s \in \Lambda \mid 1 - \mathrm{Re}\, \tau(v_s^* \varphi_n(v_s)) \leq 2\varepsilon_1 \} \; .$$

Note that for all $s \in \mathcal{V}$, we have $\|\varphi_n(v_s) - v_s\|_2 \leq \sqrt{4\varepsilon_1} = \varepsilon$. In order to prove the lemma, it suffices to show that there exists a sequence $s_k \in \mathcal{V}$ such that $\|E_P(av_{s_k}b)\|_2 \to 0$ for all $a, b \in M$. Assume the contrary. We then find a finite subset $\mathcal{F} \subset M$ and a $\delta > 0$ such that

$$\sum_{a,b \in \mathcal{F}} \|E_P(av_sb)\|_2^2 \geq \delta \quad \text{for all} \quad s \in \mathcal{V} \; .$$

We will deduce that $\Delta(pM_0p) \prec_{N \overline{\otimes} M} N \overline{\otimes} P$. This will be a contradiction with the statement $\Delta(pM_0p) \not\prec_{N \overline{\otimes} M} N \overline{\otimes} P$ that we have shown at the beginning of the proof.

Let $w \in \mathcal{U}(pM_0p)$. Write $w = \sum_{s \in \Lambda} w_s v_s$ where $w_s \in Q$. Since $\tau(p) = 1$, we have $\sum_{s \in \Lambda} \|w_s\|_2^2 = 1$. Therefore,

$$\begin{aligned}
\varepsilon_1 &\geq 1 - \mathrm{Re}(\tau \otimes \tau)(\Delta(w)^* \, (\mathrm{id} \otimes \varphi_n)\Delta(w)) \\
&= \sum_{s \in \Lambda} \|w_s\|_2^2 \left(1 - \mathrm{Re}\, \tau(v_s^* \varphi_n(v_s))\right) \\
&\geq \sum_{s \in \Lambda - \mathcal{V}} \|w_s\|_2^2 \left(1 - \mathrm{Re}\, \tau(v_s^* \varphi_n(v_s))\right) \\
&\geq \sum_{s \in \Lambda - \mathcal{V}} \|w_s\|_2^2 \, 2\varepsilon_1 \; .
\end{aligned}$$

We conclude that for all $w \in \mathcal{U}(pM_0p)$ we have

$$\sum_{s \in \Lambda - \mathcal{V}} \|w_s\|_2^2 \leq \frac{1}{2}$$

implying that

$$\sum_{s \in \mathcal{V}} \|w_s\|_2^2 \geq \frac{1}{2}$$

for all $w \in \mathcal{U}(pM_0p)$.

It follows that for all $w \in \mathcal{U}(pM_0p)$

$$\begin{aligned}\sum_{a,b\in\mathcal{F}} \|E_{N\overline{\otimes}P}((1\otimes a)\Delta(w)(1\otimes b))\|_2^2 &= \sum_{a,b\in\mathcal{F},s\in\Lambda} \|w_s\|_2^2\,\|E_P(av_sb)\|_2^2 \\ &\geq \sum_{a,b\in\mathcal{F},s\in\mathcal{V}} \|w_s\|_2^2\,\|E_P(av_sb)\|_2^2 \\ &\geq \sum_{s\in\mathcal{V}} \|w_s\|_2^2\,\delta \\ &\geq \frac{\delta}{2}\,.\end{aligned}$$

This means that $\Delta(pM_0p) \prec_{N\overline{\otimes}M} N\overline{\otimes}P$. We have reached the desired contradiction. $\square$

We are ready to formulate and prove our uniqueness of group measure space Cartan theorem. We use the notation $\mathrm{D}_n(\mathbb{C}) \subset \mathrm{M}_n(\mathbb{C})$ to denote the subalgebra of diagonal matrices.

Theorem 5. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product satisfying the following conditions: Γ_1 admits an infinite subgroup with property (T), Σ is anti-(T) and $\Gamma_2 \neq \Sigma$. Assume moreover that there exist $g_1,\ldots,g_n \in \Gamma$ such that $\bigcap_{i=1}^n g_i\Sigma g_i^{-1}$ is finite. Let $\Gamma \curvearrowright (X,\mu)$ be any free ergodic pmp action and denote $M = \mathrm{L}^\infty(X) \rtimes \Gamma$.*

Whenever $\Lambda \curvearrowright (Y,\eta)$ is a free ergodic pmp action, $p \in \mathrm{M}_n(\mathbb{C}) \otimes M$ is a projection and

$$\pi : \mathrm{L}^\infty(Y) \rtimes \Lambda \to p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$$

is a $*$-isomorphism, there exists a projection $q \in \mathrm{D}_n(\mathbb{C}) \otimes \mathrm{L}^\infty(X)$ and a unitary $u \in q(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ such that

$$\pi(\mathrm{L}^\infty(Y)) = u^*(\mathrm{D}_n(\mathbb{C}) \otimes \mathrm{L}^\infty(X))u\,.$$

Proof. Write $A = \mathrm{M}_n(\mathbb{C}) \otimes \mathrm{L}^\infty(X)$ and $N = A \rtimes \Gamma$. Put $B = \mathrm{L}^\infty(Y)$. We first prove that $\pi(B) \prec_N A \rtimes \Sigma$. Denote by $|g|$ the length of $g \in \Gamma_1 *_\Sigma \Gamma_2$ as a reduced word. Denote for $0 < \rho < 1$ by m_ρ the corresponding completely positive maps on N given by $\mathrm{m}_\rho(au_g) = \rho^{|g|}au_g$ for all $a \in A$, $g \in \Gamma$. When $\rho \to 1$, we have $\mathrm{m}_\rho \to \mathrm{id}$ pointwise in $\|\cdot\|_2$.

Write $\varepsilon = \tau(p)/5000$ and $P = A \rtimes \Sigma$. Note that P is anti-(T). By Lemma 4 take $0 < \rho_1 < 1$ and a sequence (s_k) in Λ satisfying $\|m_{\rho_1}(v_{s_k}) - v_{s_k}\|_2 \leq \varepsilon$ for all k and $\|E_P(xv_{s_k}y)\|_2 \to 0$ for all $x,y \in N$. By [PV09, Lemma 5.7] there exists a $0 < \rho < 1$ and a $\delta > 0$ such that $\tau(w^*\mathrm{m}_\rho(w)) \geq \delta$ for all $w \in \mathcal{U}(\pi(B))$. By [PV09, Theorem 5.4] and because $\pi(B)$ is regular in pNp, it follows that $\pi(B) \prec_N P$.

So we have shown that $\pi(B) \prec_N A\rtimes\Sigma$. By Proposition 8 below and since we have $g_1,\ldots,g_n \in \Gamma$ such that $\bigcap_{i=1}^n g_i\Sigma g_i^{-1}$ is finite, it follows that $\pi(B) \prec_N A$. The theorem now follows from [Po01, Theorem A.1]. $\square$

4 An embedding result, strengthening [PV06, Theorem 6.16]

Assume that (A,τ) is a tracial von Neumann algebra and that $\Gamma \overset{\sigma}{\curvearrowright} (A,\tau)$ is a trace preserving action. We do not assume that σ is properly outer or that σ is ergodic. Let $M = A \rtimes \Gamma$.

Whenever $\Lambda < \Gamma$ is a subgroup, consider the basic construction $\langle M, e_{A\rtimes\Lambda}\rangle$. By definition $\langle M, e_{A\rtimes\Lambda}\rangle$ consists of those operators on $\mathrm{L}^2(M)$ that commute with the right module action

of $A \rtimes \Lambda$. The basic construction comes with a canonical operator valued weight T_Λ from $\langle M, e_{A\rtimes\Lambda}\rangle^+$ to the extended positive part of M. For all $x, y \in M$, the element $x e_{A\rtimes\Lambda} y$ is integrable with respect to T_Λ and $T_\Lambda(x e_{A\rtimes\Lambda} y) = xy$. Choose $g_i \in \Gamma$ such that $\Gamma = \bigsqcup_i g_i\Lambda = \bigsqcup_i \Lambda g_i^{-1}$. Denoting by ρ_g the right multiplication operator by u_g^* on $\mathrm{L}^2(M)$, one checks that $\sum_i \rho_{g_i} e_{A\rtimes\Lambda} \rho_{g_i}^* = 1$, whence

$$T_\Lambda(x) = \sum_i \rho_{g_i} x \rho_{g_i}^* \quad \text{for all} \quad x \in \langle M, e_{A\rtimes\Lambda}\rangle^+ .$$

The canonical semi-finite trace Tr_Λ on $\langle M, e_{A\rtimes\Lambda}\rangle$ is given as the composition of T_Λ and the trace τ on M.

Assume that $p \in M$ is a projection and $B \subset pMp$ is a quasi-regular subalgebra (see [Po01, 1.4.2]). Recall that this means that the quasi-normalizer of B inside pMp is weakly dense in pMp. Obviously, regular subalgebras $B \subset pMp$ or, even more specifically, Cartan subalgebras $B \subset pMp$ are quasi-regular.

Given a subgroup $\Lambda < \Gamma$, let $H \subset p\,\mathrm{L}^2(M)$ be the closed linear span of all B-$(A \rtimes \Lambda)$-subbimodules of $p\,\mathrm{L}^2(M)$ that are finitely generated as a right Hilbert $(A \rtimes \Lambda)$-module. Since $B \subset pMp$ is quasi-regular, H is stable by left multiplication with pMp. The subspace H is also invariant under right multiplication by $A \rtimes \Lambda$. So H is of the form $p\,\mathrm{L}^2(M)z(\Lambda)$ for some projection $z(\Lambda) \in M \cap (A \rtimes \Lambda)'$. We make $z(\Lambda)$ uniquely defined by requiring that $z(\Lambda)$ is smaller or equal than the central support of p in M.

Note that by definition $z(\Lambda) \neq 0$ if and only if $B \prec_M A \rtimes \Lambda$.

Denote by $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ the adjoint operator. So, given $x \in M$, Jx^*J is the operator of right multiplication with x. Denote by $\operatorname{supp} a$ the support projection of a positive operator a. Observe that

$$\begin{aligned}
&pJz(\Lambda)J \\
&\quad= \bigvee \{q \mid q \ \text{ is an orthogonal projection in } B' \cap p\langle M, e_{A\rtimes\Lambda}\rangle p \text{ satisfying } \ \mathrm{Tr}_\Lambda(q) < \infty\} \\
&\quad= \bigvee \{q \mid q \ \text{ is an orthogonal projection in } B' \cap p\langle M, e_{A\rtimes\Lambda}\rangle p \text{ satisfying } \ \|T_\Lambda(q)\| < \infty\} \\
&\quad= \bigvee \{\operatorname{supp} a \mid a \in (B' \cap p\langle M, e_{A\rtimes\Lambda}\rangle p)^+ \ \text{ and } \ \|T_\Lambda(a)\| < \infty\} .
\end{aligned} \tag{1}$$

If a and b are positive operators, then $\operatorname{supp}(a) \vee \operatorname{supp}(b) = \operatorname{supp}(a+b)$. So we find a sequence of elements $a_n \in (B' \cap p\langle M, e_{A\rtimes\Lambda}\rangle p)^+$ such that all $T_\Lambda(a_n)$ are bounded and $\operatorname{supp}(a_n) \to pJz(\Lambda)J$ strongly. Moreover, every projection $\operatorname{supp}(a_n)$ can be strongly approximated by a spectral projection of the form $q_n = \chi_{[\varepsilon_n, +\infty)}(a_n)$ for $\varepsilon_n > 0$ sufficiently small. We have $q_n \leq \frac{1}{\varepsilon_n} a_n$ so that $T_\Lambda(q_n)$ is bounded. Hence we find a sequence of projections $q_n \in B' \cap p\langle M, e_{A\rtimes\Lambda}\rangle p$ such that $q_n \to pJz(\Lambda)J$ strongly and $\|T_\Lambda(q_n)\| < \infty$ for all n.

Note that $z(\Lambda_1) \leq z(\Lambda_2)$ when $\Lambda_1 < \Lambda_2 < \Gamma$. Indeed, it suffices to observe that for every B-$(A \rtimes \Lambda_1)$-subbimodule $K \subset p\,\mathrm{L}^2(M)$ that is finitely generated as a right Hilbert module, the closed linear span of $K(A \rtimes \Lambda_2)$ is a B-$(A \rtimes \Lambda_2)$-subbimodule of $p\,\mathrm{L}^2(M)$ that is finitely generated as a right Hilbert module. Hence $p\,\mathrm{L}^2(M)z(\Lambda_1) \subset p\,\mathrm{L}^2(M)z(\Lambda_2)$. Since by convention $z(\Lambda_1)$ and $z(\Lambda_2)$ are smaller or equal than the central support of p, we conclude that $z(\Lambda_1) \leq z(\Lambda_2)$.

By definition, $z(g\Lambda g^{-1}) = u_g z(\Lambda) u_g^*$ for all $g \in \Gamma$, $\Lambda < \Gamma$.

Proposition 6. *Let $M = A \rtimes \Gamma$ for some trace preserving action of a countable group Γ on the tracial von Neumann algebra (A, τ). Let $B \subset pMp$ be a quasi-regular von Neumann subalgebra. For every subgroup $\Lambda < \Gamma$, define as above the projection $z(\Lambda) \in M \cap (A \rtimes \Lambda)'$ such that*

$p\,\mathrm{L}^2(M)z(\Lambda)$ equals the closed linear span of all B-$(A \rtimes \Lambda)$-subbimodules of $p\,\mathrm{L}^2(M)$ that are finitely generated as a right Hilbert module.

If $\Sigma < \Gamma$ and $\Lambda < \Gamma$ are subgroups, then the projections $z(\Sigma)$ and $z(\Lambda)$ commute and satisfy

$$z(\Sigma \cap \Lambda) = z(\Sigma)\, z(\Lambda)\;.$$

Proof. We use the operator valued weight T_Σ as explained above. As we saw after formulae (1) we can take projections $q_n \in B' \cap p\langle M, e_{A\rtimes\Sigma}\rangle p$ and $e_n \in B' \cap p\langle M, e_{A\rtimes\Lambda}\rangle p$ such that $q_n \to pJz(\Sigma)J$ and $e_n \to pJz(\Lambda)J$ strongly and such that for all $n \in \mathbb{N}$, we have $\|T_\Sigma(q_n)\| < \infty$ and $\|T_\Lambda(e_n)\| < \infty$. Note that $e_n q_n e_n \in (B' \cap p\langle M, e_{A\rtimes(\Sigma\cap\Lambda)}\rangle p)^+$. We claim that

$$\|T_{\Sigma\cap\Lambda}(e_n q_n e_n)\| < \infty \quad \text{for all } \; n \in \mathbb{N}\;. \tag{2}$$

Fix $n \in \mathbb{N}$. Take $g_i \in \Gamma$ such that $\Gamma = \bigsqcup_i g_i \Lambda$. Take $h_j \in \Lambda$ such that $\Lambda = \bigsqcup_j h_j(\Sigma \cap \Lambda)$. Note that the cosets $h_j\Sigma$ are disjoint and that $\Gamma = \bigsqcup_{i,j} g_i h_j(\Sigma \cap \Lambda)$. Because of the latter, we have

$$T_{\Sigma\cap\Lambda}(e_n q_n e_n) = \sum_{i,j} \rho_{g_i h_j} e_n q_n e_n \rho^*_{g_i h_j} = \sum_i \rho_{g_i} e_n \Bigl(\sum_j \rho_{h_j} q_n \rho^*_{h_j}\Bigr) e_n \rho^*_{g_i}\;.$$

Because the cosets $h_j\Sigma$ are disjoint, we know that

$$\sum_j \rho_{h_j} q_n \rho^*_{h_j} \leq \sum_{h \in \Gamma/\Sigma} \rho_h q_n \rho^*_h = T_\Sigma(q_n) \leq \|T_\Sigma(q_n)\|\, 1\;.$$

Therefore,

$$T_{\Sigma\cap\Lambda}(e_n q_n e_n) \leq \|T_\Sigma(q_n)\| \sum_i \rho_{g_i} e_n \rho^*_{g_i} = \|T_\Sigma(q_n)\|\, T_\Lambda(e_n)\;.$$

So claim (2) is proven.

Since $e_n q_n e_n \in (B' \cap p\langle M, e_{A\rtimes(\Sigma\cap\Lambda)}\rangle p)^+$ claim (2) implies that

$$\operatorname{supp}(e_n q_n e_n) \leq pJz(\Sigma \cap \Lambda)J \quad \text{for all } \; n \in \mathbb{N}\;.$$

Hence, $\operatorname{Ran}(e_n q_n) \subset p\,\mathrm{L}^2(M)z(\Sigma \cap \Lambda)$ for all n. Since $e_n \to pJz(\Lambda)J$ and $q_n \to pJz(\Sigma)J$ strongly and since $z(\Lambda)$ and $z(\Sigma)$ are chosen below the central support of p, it follows that

$$\operatorname{Ran}(z(\Lambda)z(\Sigma)) \subset \operatorname{Ran} z(\Sigma \cap \Lambda)\;.$$

Since $z(\Sigma \cap \Lambda) \leq z(\Sigma)$ and $z(\Sigma \cap \Lambda) \leq z(\Lambda)$, we get the chain of inclusions

$$\operatorname{Ran} z(\Lambda) \cap \operatorname{Ran} z(\Sigma) \subset \operatorname{Ran}(z(\Lambda)z(\Sigma)) \subset \operatorname{Ran} z(\Sigma \cap \Lambda) \subset \operatorname{Ran} z(\Lambda) \cap \operatorname{Ran} z(\Sigma)\;.$$

So all these inclusions are equalities. This means that $z(\Lambda)$ and $z(\Sigma)$ are commuting projections and $z(\Sigma \cap \Lambda) = z(\Sigma)\, z(\Lambda)$. □

As an immediate corollary we find the following generalization of [PV06, Theorem 6.16].

Corollary 7. *Let $\Gamma \curvearrowright A$ be a trace preserving action, $M = A \rtimes \Gamma$, $\Sigma < \Gamma$ a subgroup and $B \subset pMp$ a quasi-regular von Neumann subalgebra. If $z(\Sigma)$ equals the central support of p, then the same is true for $z(g_1\Sigma g_1^{-1} \cap \cdots \cap g_n \Sigma g_n^{-1})$ and in particular*

$$B \prec_M A \rtimes (g_1\Sigma g_1^{-1} \cap \cdots \cap g_n \Sigma g_n^{-1}) \tag{3}$$

for all $g_1, \ldots, g_n \in \Gamma$

5 Application to the proofs of Theorem 5, [PV09, Theorems 5.2 and 1.4] and [FV10, Theorem 4.1]

In the setup of Theorem 5 and [PV09, Theorems 5.2 and 1.4] we know that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is an amalgamated free product and we know that $A \rtimes \Gamma$ is a factor. We prove that in such a situation, whenever $B \subset p(A \rtimes \Gamma)p$ is a quasi-regular subalgebra, $z(\Sigma)$ can only take the values 0 or 1. A similar statement is true when $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ is an HNN extension(7) of a countable group H with a subgroup $\Sigma < H$ and an injective group homomorphism $\theta : \Sigma \to H$.

The precise formulation goes as follows.

Proposition 8. *Let Γ either be an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ with $\Gamma_1 \neq \Sigma \neq \Gamma_2$ or an arbitrary HNN extension $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$. Let (A, τ) be a tracial von Neumann algebra and $\Gamma \curvearrowright A$ a trace preserving action. Put $M = A \rtimes \Gamma$ and let $B \subset pMp$ be a quasi-regular von Neumann subalgebra. As above we define for every subgroup $\Lambda < \Gamma$, the projection $z(\Lambda) \in M \cap (A \rtimes \Lambda)'$ such that $p\,\mathrm{L}^2(M)z(\Lambda)$ is the closed linear span of all B-$(A \rtimes \Lambda)$-subbimodules of $p\,\mathrm{L}^2(M)$ that are finitely generated as a right Hilbert module.*

The projection $z(\Sigma)$ belongs to the center of M. So, if M is a factor and if $B \prec_M A \rtimes \Sigma$, then $z(\Sigma) = 1$ and

$$B \prec_M A \rtimes (g_1 \Sigma g_1^{-1} \cap \cdots \cap g_n \Sigma g_n^{-1})$$

for all $g_1, \ldots, g_n \in \Gamma$.

Proof. Assume first that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is a non-trivial amalgamated free product. We use Proposition 6 to prove that $z(\Sigma) = z(\Gamma_1)$. Once this is proven, by symmetry also $z(\Sigma) = z(\Gamma_2)$. But then $z(\Sigma)$ commutes with both $A \rtimes \Gamma_1$ and $A \rtimes \Gamma_2$, so that $z(\Sigma)$ belongs to the center of M.

Put $z_1 = z(\Gamma_1)$. Define $S \subset \Gamma$ as the set of elements $g \in \Gamma$ that admit a reduced expression starting with a letter from $\Gamma_2 - \Sigma$. Whenever $g \in S$, we have $g\Gamma_1 g^{-1} \cap \Gamma_1 \subset \Sigma$. It follows from Proposition 6 that the projections $z_1 = z(\Gamma_1)$ and $u_g z_1 u_g^* = z(g\Gamma_1 g^{-1})$ commute and that

$$z(\Sigma) \geq z(g\Gamma_1 g^{-1} \cap \Gamma_1) = u_g z_1 u_g^* \, z_1 \quad \text{for all } \ g \in S \; . \tag{4}$$

We claim that

$$z_1 = \bigvee_{g \in S} u_g z_1 u_g^* \, z_1 \quad .$$

To prove this claim, put

$$q = z_1 - \bigvee_{g \in S} u_g z_1 u_g^* \, z_1 \quad .$$

Whenever $g \in S$, we have

$$q \, u_g q u_g^* = q z_1 \, u_g z_1 q u_g^* = q \, z_1 \, u_g z_1 u_g^* \, u_g q u_g^* = q \, u_g z_1 u_g^* \, z_1 \, u_g q u_g^* = 0 \; .$$

Take $a \in \Gamma_1 - \Sigma$ and $b \in \Gamma_2 - \Sigma$ and put $u_n = u_{(ba)^n}$. It follows that the projections $u_n q u_n^*$, $n \in \mathbb{N}$, are mutually orthogonal. Indeed, if $n < m$, we have

$$u_n q u_n^* \, u_m q u_m^* = u_n \big(q \, u_{m-n} q u_{m-n}^*\big) u_n^* = 0$$

(7) The HNN extension $\mathrm{HNN}(H, \Sigma, \theta)$ is generated by a copy of H and an element t, called stable letter, subject to the relation $t\sigma t^{-1} = \theta(\sigma)$ for all $\sigma \in \Sigma$.

because $(ba)^{m-n} \in S$. Since τ is a finite trace on M, it follows that $\tau(q) = 0$ and hence, $q = 0$. This proves the claim. In combination with (4) it follows that $z(\Sigma) \geq z_1$. Hence, $z(\Sigma) = z_1$.

Next assume that $\Gamma = \text{HNN}(H, \Sigma, \theta)$. Denote by $t \in \Gamma$ the stable letter. For every $n \geq 1$, one has $t^{-n} H t^n \cap H \subset \Sigma$. The same argument as in the case of amalgamated free products shows that $z(\Sigma) = z(H)$. We also have $\Sigma \subset t^{-1} H t$. Hence $z(\Sigma) \leq z(t^{-1} H t)$. Since $z(H) = z(\Sigma)$ and $z(t^{-1} H t) = u_t^* z(H) u_t$, we conclude that $z(H) \leq u_t^* z(H) u_t$. The left and right hand side have the same trace and hence must be equal. It follows that $z(\Sigma)$ commutes with u_t. Since $z(\Sigma)$ already commutes with $A \rtimes H$, it follows that $z(\Sigma)$ belongs to the center of M.

Finally consider the special case where M is a factor and $B \prec_M A \rtimes \Sigma$. The latter precisely means that $z(\Sigma) \neq 0$. Since $z(\Sigma)$ is a projection in the center of M, it follows that $z(\Sigma) = 1$. Then also $z(g \Sigma g^{-1}) = 1$ for all $g \in \Gamma$. By Proposition 6, we get that

$$z(g_1 \Sigma g_1^{-1} \cap \cdots \cap g_n \Sigma g_n^{-1}) = 1$$

for all $g_1, \ldots, g_n \in \Gamma$. In particular,

$$B \prec_M A \rtimes (g_1 \Sigma g_1^{-1} \cap \cdots \cap g_n \Sigma g_n^{-1})$$

for all $g_1, \ldots, g_n \in \Gamma$. □

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